Dedicated to John T.Lewis

Bose-Einstein Condensation in the Luttinger-Sy Model

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Abstract

We present a rigorous study of the Bose-Einstein condensation in the *Luttinger-Sy* model. We prove the existence of the condensation in this one-dimensional model of the perfect boson gas placed in the Poisson random potential of singular point impurities. To tackle the off-diagonal long-range order we calculate explicitly the corresponding space-averaged one-body reduced density matrix. We show that mathematical mechanism of the Bose-Einstein condensation in this random model is similar to condensation in a one-dimensional *nonrandom hierarchical* model of scaled intervals. For the Luttinger-Sy model we prove the *Kac-Luttinger conjecture*, i.e., that this model manifests a *type* I BEC localized in a single "largest" interval of *logarithmic* size.

Keywords: Generalized Bose-Einstein condensation, random potential, density of states, Lifshitz tail, Kac-Luttinger conjecture

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1 Introduction

1.1 In our recent paper [13], we presented some general mathematical results concerning the existence of the *Bose-Einstein condensation* (BEC) of the *Perfect Bose-Gas* (PBG) placed in a semi-bounded from below homogeneous ergodic random external potential (random impurities). There we show that for the infinite-volume one-particle Schrödinger operator, a generic *Lifshitz tail* behaviour of density of states near the lower edge of the spectrum reduces the *critical dimensionality* of the BEC for PBG from dimensionality $d = 2 + \varepsilon$ to $d = 1$. Therefore, the randomness *enhances* the BEC, and moreover, it is shown to be stable with respect to the *mean-field* particle interaction [12].

To tackle the corresponding *Off-Diagonal Long-Range Order* (ODLRO), we introduced in [13] a concept of the *space-averaged* one-body reduced density matrix. In spite of a rather accurate estimate of this matrix *out* of the BEC domain, that shows an enhancement of the ODLRO exponential *decay* due to impurities, we did not obtain in [13] any sound estimate for this order *in* the BEC domain.

1.2 The aim of this paper is to present a rigorous study of a particular case of a one-dimensional PBG model in homogeneous ergodic non-negative random potential induced by the Poisson distributed *singular point* impurities (the *Luttinger-Sy* model [17, 18]). We show that this model allows a rigorous mathematical approach to condensation and that one can compute explicitly some of thermodynamical quantities even in the BEC domain. This concerns in particular the ODLRO behaviour of the two-point correlation function (space-averaged one-body reduced density matrix) in the condensation regime.

Notice that the first study of possible modifications of $d = 3$ dimensional BEC in the PBG caused by repulsive finite-range impurities goes back to Kac and Luttinger [8, 9]. They predicted an enhancement of $d = 3$ BEC by indication that due to impurities there is decreasing of the critical density , but did not discuss a modification of the critical dimensionality. They also mentioned a puzzling question about the nature of the established BEC. For example, they *conjectured* that this condensate occurs as a macroscopic occupation of only the ground-state: *type I* BEC. We prove this conjecture in the case of the Luttinger-Sy model, see discussion in Section 6. We show that the nature of BEC in this model is close to what is known as the "*Bose-glass*", since it may be localized by the random potential. This is of interest for example in experiments with liquid ^4He in random environments like Aerogel and Vycor glasses, [6, 10].

On the other hand, the nature and behaviour of the *lattice* BEC may be quite different. First of all, the lattice Laplacian and the on-site *Bose-Hubbard* particle repulsion produces a coexistence of the BEC (*superfluidity*) and the *Mott insulating phase* as well as domains of *incompressibility*, see e.g. a very complete review [23]. Adding disorder makes the corresponding models much more complicated. The physical arguments show that the randomness may *suppress* the BEC (superfluidity) as well as the Mott phase in favour of the localized *Bose-glass* phase, but this is very sensitive to the choice of the random distribution, for some recent rigorous results see [5].

1.3 The paper is organized as follows. In Section 2 we recall definition of the Luttinger-Sy model and some of its properties. We prove the *self-averaging* of corresponding *integrated density of states* in Section 3 and we calculate it explicitly. In Section 4 we prove that the established integrated density of states implies the existence of *generalized* BEC in the case of PBG.

Our main results are collected in Sections 5 and 6. There we recall the notion of the *space-averaged* one-body reduced density matrix and we prove that it has an *almost sure* nonrandom thermodynamic limit (self-averaging), which can be calculated explicitly for all values of particle density. We also prove that randomness *enhances* decay of the two-point correlation function. In particular we show that it keeps this decay always *exponential*, even in the presence of BEC. We found that in the latter case the ODLRO is non-zero and that it coincides with the condensate density.

The properties of the BEC are discussed in concluding Section 6. First, we analyze the critical density dependence on the amplitude of the repulsive Poisson point impurities. Notice that for the Luttinger-Sy model the singular point impurities mean that this amplitude is infinite. Next, we study the problem of the condensate nature and its *localization*. To elucidate this point we invented a *hierarchical* one-dimensional *nonrandom* model, which mimics in a certain sense the (random) Luttinger-Sy model. We show that this hierarchical model can manifest different *types* (I,II and III) of *generalized* van den Berg-Lewis-Pulé condensations [1] localized in one, several or infinite number of (*infinite*) intervals of *logarithmic* sizes. To discriminate between these options, i.e. to prove or disprove the Kac-Luttinger *conjecture*, one has to have a quite detailed information about the energy level spacing in random intervals generated by the Poisson impurity positions. We prove this conjecture, i.e., that *type* I BEC in the Luttinger-Sy model is localized in a single "largest" (i.e. infinite) interval of the *logarithmic* size.

2 The Luttinger-Sy Model

In the framework of general setting this model corresponds to the following one-dimensional ($d = 1$) single-particle *random* Schrödinger operator in the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$:

2.1 Consider a random (measurable) potential $v^{(\cdot)}(\cdot) : \Omega \times \mathbb{R} \to \mathbb{R}, (\omega, x) \mapsto v^{\omega}(x)$, which is a random field on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with the properties:

(a) v^{ω} is homogeneous and ergodic with respect to the group $\{\tau_x\}_{x\in\mathbb{R}}$ of probability-preserving translations on $(\Omega, \mathcal{F}, \mathbb{P})$;

(b) v^{ω} is non-negative and $\inf_{x\in\mathbb{R}^d} \{v^{\omega}(x)\} = 0$.

By $\mathbb{E}\{\cdot\} := \int_{\Omega} \mathbb{P}(d\omega)\{\cdot\}$ we denote the expectation with respect to the probability measure in $(\Omega, \mathcal{F}, \mathbb{P})$. Then the *random* Schrödinger operator corresponding to the potential v^{ω} is a family of random operators $\{h^{\omega}\}_{\omega\in\Omega}$:

$$h^{\omega} := t + v^{\omega}, \tag{2.1}$$

where $t := (-\Delta/2)$ is the *free* one-particle Hamiltonian, i.e., a unique self-adjoint extension of the operator: $-\Delta/2$, with domain in $L^2(\mathbb{R})$.

Notice that assumptions (a) and (b) guarantee that there exists a subset $\Omega_0 \subset \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that operator (2.1) is *essentially* self-adjoint on domain $\mathcal{C}_0^{\infty}(\mathbb{R})$ for every $\omega \in \Omega_0$ (see e.g. [20] Ch.I.2).

2.2 Let $u(x) \geq 0$, $x \in \mathbb{R}$, be continuous function with a *compact* support. We call it a (*repulsive*) single-impurity potential. Let $\{\nu_{\lambda}^{\omega}(dx)\}_{\omega\in\Omega}$ be *random* Poisson measure on $\mathbb{R}$ with intensity $\lambda > 0$:

$$\mathbb{P}\left(\{\omega \in \Omega : \nu_{\lambda}^{\omega}(\Lambda) = n\}\right) = \frac{(\lambda\,|\Lambda|)^n}{n!} e^{-\lambda|\Lambda|} \;\;,\;\; n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}\,, \tag{2.2}$$

for any bounded Borel set $\Lambda \subset \mathbb{R}$. Then the non-negative random potential v^{ω} generated by the Poisson distributed local impurities has realizations

$$v^{\omega}(x) := \int_{\mathbb{R}} \nu_{\lambda}^{\omega}(dy) u(x - y) = \sum_{x_j^{\omega}\in X^{\omega}} u(x - x_j^{\omega}). \tag{2.3}$$

Here the random set X^ω corresponds to impurity positions $X^\omega = \left\{x_j^\omega\right\}_j \subset \mathbb{R}$, which are the atoms of the random Poisson measure, i.e., *card* $\{X^\omega \upharpoonright \Lambda\} = \nu_\lambda^\omega(\Lambda)$ equals to the number of impurities in the set Λ. Since the expectation $\mathbb{E}\left(\nu_\lambda^\omega(\Lambda)\right) = \lambda\,|\Lambda|$, the parameter λ coincides with impurities *concentration* on the axe $\mathbb{R}$.

Remark 2.1 *The random potential (2.3) is obviously homogeneous and ergodic (even strongly mixing), i.e. it verifies the conditions (a) and (b). Moreover, see [20] Ch.II.5, we have that:*
- There exists a nonrandom measure $d\mathcal{N}(E)$ on $\mathbb{R}$ such that

$$d\mathcal{N}(E) := \mathbb{E}\left\{\mathcal{E}_{h^\omega}(dE;0,0)\right\}. \tag{2.4}$$

Here $\mathcal{E}_{h^\omega}(dE;x,y)$ is the kernel of the spectral decomposition measure corresponding to random Schrödinger operator h^ω. The spectrum $\sigma(h^\omega)$ of h^ω is almost-surely (a.s.) *nonrandom and it coincides with the support of $\mathcal{N}$: $\sigma(h^\omega) = \operatorname{supp}\mathcal{N}$.*
- For repulsive impurities with compact support and for Poisson distribution, the a.s nonrandom spectrum $\sigma(h^\omega) = \mathbb{R}_+$. Thus the lower edge of the spectrum $\inf\{\sigma(h^\omega)\} = 0$*, i.e. it coincides with the lower edge of the spectrum of the nonrandom operator t, see (2.1).*
- In one-dimensional case the asymptotic behaviour of the integrated density of states $\mathcal{N}(E) := \mathcal{N}((-\infty, E])$ as $E \downarrow 0$ has the form (the Lifshitz tail):

$$\ln \mathcal{N}(E) \sim -\lambda \left(\frac{c_d}{E}\right)^{d/2}\ ,\ E \downarrow 0\ , \tag{2.5}$$

for $c_d > 0$. Recall that in the nonrandom case $v^\omega = 0$ one obtains: $\mathcal{N}(E) \sim E^{d/2},\ E \downarrow 0$.

2.3 Luttinger and Sy defined their $d = 1$ model [17] restricting the single-impurity potential to the *point* δ-potential with amplitude $a > 0$. In fact this choice (even for more general case of random $\{a_j\}_j$) goes back to Frish and Lloyd [7]. Then the corresponding random potential (2.3) takes the form:

$$v_a^\omega(x) := \int_{\mathbb{R}} \nu_\lambda^\omega(dy) a\delta(x-y) = a \sum_{x_j^\omega \in X^\omega} \delta(x - x_j^\omega). \tag{2.6}$$

Now the self-adjoint one-particle random Schrödinger operator

$$h_a^\omega := t \dot{+} v_a^\omega, \tag{2.7}$$

can be defined in the sense of the sum of quadratic forms. In spite of a singular nature of this random potential, by standard limiting arguments [20] it inherits the properties quoted in Remark 2.1.

2.4 Moreover, the same arguments [20] are applied to define a *strong resolvent* (s.r.) limit of Hamiltonians (2.7), when $a \to +\infty$, which is the *last step* in definition of the Luttinger-Sy model, [17]. This limit gives the self-adjoint (*Friedrichs*) extension of symmetric operator $t_0 = -\Delta/2$ with domain $dom(t_0) = \{f \in \mathcal{H} : f \in \mathcal{C}_0^\infty(\mathbb{R} \setminus X^\omega)\}$. For any $\omega \in \Omega$ we denote this extension by

$$h_D^\omega := s.r.\lim_{a\to+\infty} h_a^\omega\ . \tag{2.8}$$

Since for any $\omega \in \Omega$ the set X^ω can be *ordered*: $X^\omega = \left\{x_j^\omega\right\}_j$, it generates a set of intervals $\left\{I_j^\omega := (x_{j-1}^\omega, x_j^\omega)\right\}_j$ of lengths $\left\{L_j^\omega := x_j^\omega - x_{j-1}^\omega\right\}_j$. Then one can decompose the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ into (random) direct orthogonal sum:

$$\mathcal{H} = \bigoplus_j \mathcal{H}_j\ ,\ \ \mathcal{H}_j := L^2(I_j^\omega). \tag{2.9}$$

Correspondingly, let $h_D(I_j^\omega)$ denote the *Friedrichs* extension of operator $t_0 = -\Delta/2$ with domain $\mathrm{dom}(t_0) = \{f \in L^2(I_j^\omega) : f \in \mathcal{C}_0^\infty(I_j^\omega)\}$:

$$(h_D(I_j^\omega)f)(x) := -\frac{1}{2}(\Delta f)(x)\,, \tag{2.10}$$
$$f \in \mathrm{dom}(h_D(I_j^\omega)) = \left\{f \in W_2^2(I_j^\omega) : f(x_{j-1}^\omega) = f(x_j^\omega) = 0\right\},$$

where W_2^2 denotes the corresponding Sobolev space. Then we get decompositions of the one-particle Luttinger-Sy Hamiltonian:

$$h_D^\omega = \bigoplus_j h_D(I_j^\omega)\ , \quad \omega \in \Omega\ , \tag{2.11}$$

with domain

$$\mathrm{dom}(h_D^\omega) = \bigoplus_j \mathrm{dom}(h_D(I_j^\omega)) \subset \mathcal{H}\ , \tag{2.12}$$

into random disjoint free Schrödinger operators $\left\{h_D(I_j^\omega)\right\}_{j,\omega}$ with *Dirichlet* boundary conditions at the end-points of the intervals $\left\{I_j^\omega\right\}_j$. The corresponding eigenfunctions have the form:

$$\Psi_{s_j,D}^\omega(x) = (0, 0, \ldots, \psi_{j,s_j}^\omega(x), 0, \ldots)\ , \tag{2.13}$$

with eigenvalues $\left\{E_{s_j}(L_j^\omega)\right\}_{s_j}$:

$$h_D^\omega\ \Psi_{s_j,D}^\omega = E_{s_j}(L_j^\omega)\ \Psi_{s_j,D}^\omega\ . \tag{2.14}$$

Remark 2.2 *For a given realization $\omega \in \Omega$ the spectrum of the random operator (2.10) is explicitly defined by non-degenerate eigenvalues*

$$\sigma(h_D(I_j^\omega)) = \left\{E_{s_j}(L_j^\omega) = \frac{1}{2}\frac{\pi^2 s_j^2}{(L_j^\omega)^2}\right\}_{s_j=1}^{\infty}\ , \tag{2.15}$$

with the corresponding eigenfunctions

$$\psi_{j,s_j}^\omega(x) = \mathbb{I}_{I_j^\omega}(x)\ \sqrt{\frac{2}{L_j^\omega}}\sin\left(\frac{\pi s_j}{L_j^\omega}(x - x_{j-1}^\omega)\right)\ . \tag{2.16}$$

Here $\mathbb{I}_{I_j^\omega}(x)$ is the characteristic function of the interval I_j^ω. By consequence, the spectrum of the Luttinger-Sy Hamiltonian (2.11) is the union of (2.15)

$$\sigma(h_D^\omega) = \bigcup_j \sigma(h_D(I_j^\omega)). \tag{2.17}$$

By virtue of Remark 2.1 this spectrum is a.s. *nonrandom*, and it coincides with support of the integrated density of states $\mathcal{N}$. Moreover, in the case of the Luttinger-Sy Hamiltonian (2.11) it is known explicitly since [17]. But the rigorous study and in particular the concept of "self-averaging", which ensures this nonrandom property, are due to [16].

3 Self-averaging of the integrated density of states

For the reader convenience we recall in this section some arguments that one uses to derive the spectral properties of the Luttinger-Sy one-particle Hamiltonian. Since our aim is to study the thermodynamic properties and Bose-Einstein condensation in this model, it is useful to derive the integrated density of states first for a *finite* system.

3.1 Let $\Lambda := [-L/2, L/2] \subset \mathbb{R}$. Then the *finite* Luttinger-Sy model with the Dirichlet boundary conditions at $x = \pm L/2$ and with $n-1$ singular point-repulsive ($a \to +\infty$) impurities corresponds to the one-dimensional self-adjoint Schrödinger operator

$$h_{L,X_n} := \bigoplus_{j=1}^{n} h_D(I_j) \ , \tag{3.1}$$

acting in the direct orthogonal sum of Hilbert spaces (2.9):

$$\mathcal{H}_\Lambda := \bigoplus_{j=1}^{n} \mathcal{H}_j \ . \tag{3.2}$$

Here

$$X_n = \{x_0 = -L/2 < x_1 < x_2 < \ldots < x_{n-1} < x_n = L/2\} \ , \ \{I_j = (x_{j-1}, x_j)\}_{j=1}^{n} \ , \tag{3.3}$$

and operators $\{h_D(I_j)\}_{j=1}^{n}$ are defined by (2.10).

To make this system *disordered*, Luttinger and Sy [17] supposed that the impurity positions are random variables, which are *independently* and *uniformly* distributed over the interval Λ. Then instead of (3.3) one gets the random sets $\{X_n^\omega\}_{\omega\in\Omega}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, which a.s. contain $n-1$ ordered impurities $\left\{x_j^\omega\right\}_{j=1}^{n-1}$. We denote the corresponding random Luttinger-Sy Hamiltonian and eigenfunctions in Λ by

$$h_{D,n,L}^\omega := h_{L,X_n^\omega} = \bigoplus_{j=1}^{n} h_D(I_j^\omega) \ , \quad h_{D,n,L}^\omega \Psi_{s_j,D,n}^{L,\omega} = E_{s_j}(L_j^\omega)\Psi_{s_j,D,n}^{L,\omega} \ , \tag{3.4}$$

where the eigenfunctions have the form:

$$\Psi_{s_j,D,n}^{L,\omega} = (0, 0, \ldots, \psi_{j,s_j}^\omega(x), \ldots, 0) \in \bigoplus_{j=1}^{n} \mathcal{H}_j \ , \tag{3.5}$$

see definitions (2.11)-(2.14).

Notice that Remark 2.2 is valid in this case modulo the substitution of h_D^ω by the random operator (3.4). In particular, for the spectrum of (3.4) one gets representation:

$$\sigma(h_{D,n,L}^\omega) = \bigcup_{j=1}^{n} \sigma(h_D(I_j^\omega)). \tag{3.6}$$

The following proposition is an immediate consequence of the hypothesis about the independent uniform impurities distribution and the *thermodynamic limit*: $L \to \infty, n \to \infty$, with a fixed density of impurities

$$\lambda = \lim_{L\to\infty} \frac{n}{L} \ . \tag{3.7}$$

Proposition 3.1 (a) *In the thermodynamic limit the above finite-volume random point field $\{X_n^\omega\}$ converges (in distribution) to the Poisson point field $\{X^\omega\}$ with intensity λ and corresponding random Poisson measure (2.2).*
(b) *The uniform and independent distribution of $n-1$ points of impurities induces on Λ a random sets of intervals $\{I_j^\omega\}_{j=1}^n, \omega \in \Omega$, of random lengths $\{L_j = L_j^\omega\}_{j=1}^n$. The corresponding joint probability distribution is*

$$dP_{L,n}(L_1, ..., L_n) = \frac{(n-1)!}{L^{n-1}} \, \delta(L_1 + ... + L_n - L) \ \ dL_1 dL_2 \dots dL_n \ . \tag{3.8}$$

(c) *In the thermodynamic limit the lengths $\{L_j^\omega\}_j$ form an infinite set of independent random variables and distribution corresponding to (3.8) converges (weakly) to the product-measure distribution σ_λ defined by the set of consistent marginals:*

$$d\sigma_{\lambda,k}(L_{j_1}, \dots, L_{j_k}) = \lambda^k \prod_{s=1}^{k} e^{-\lambda L_{j_s}} dL_{j_s} \ . \tag{3.9}$$

The proof is standard [2, 22], see e.g. [12] for details.

Recall that the finite-volume *integrated density of states* is defined by specific *counting-function* [20]. For operator (3.4), it is a *random* variable of the form:

$$\mathcal{N}_L^\omega(E) := \frac{1}{L} \sum_{\{\Psi_{s,D,n}^{L,\omega}\}} \theta(E - E_{s,D}^\omega(n, L)) = \frac{1}{L} \int_{-L/2}^{L/2} dx \ \theta(E - h_{D,n,L}^\omega)(x, x) \ . \tag{3.10}$$

Here $\theta(E - h_{D,n,L}^\omega)(x, y)$ is the kernel of the spectral-projection operator of $h_{D,n,L}^\omega$ corresponding to the half-line $(-\infty, E)$ and $\theta(x) = \mathbb{I}_{(0,+\infty)}(x)$ stands for the *step-function*.

Proposition 3.2 *In thermodynamic limit the finite-volume integrated density of states (3.10) converges a.s. to non-random function*

$$\mathcal{N}_\lambda(E) := \lambda \ \frac{e^{-c\lambda/\sqrt{E}}}{1 - e^{-c\lambda/\sqrt{E}}} \ , \tag{3.11}$$

with $c = \pi/\sqrt{2}$.

Proof: Explicit expressions (2.15) and (2.16) imply for (3.10) the representation:

$$\mathcal{N}_L^\omega(E) = \frac{1}{L} \sum_{j=1}^{n} \sum_{s=1}^{\infty} \theta \left\{ E - \left(\frac{cs}{L_j^\omega} \right)^2 \right\} \ . \tag{3.12}$$

Then by Proposition 3.1 and by (3.7), (3.12) we obtain

$$\mathcal{N}_\lambda(E) := a.s. \lim_{L\to\infty} \mathcal{N}_L^\omega(E) = a.s. \lim_{n\to\infty} \frac{\lambda}{n} \sum_{j=1}^{n} \sum_{s=1}^{\infty} \theta \left\{ E - \left(\frac{cs}{L_j^\omega} \right)^2 \right\} =$$

$$\lambda \ \mathbb{E}_{\sigma_\lambda} \left\{ \sum_{s=1}^{\infty} \theta \left(E - \left(\frac{cs}{L_i} \right)^2 \right) \right\} = \lambda^2 \sum_{s=1}^{\infty} \int_0^\infty dL_i e^{-\lambda L_i} \theta \left(E - \left(\frac{cs}{L_i} \right)^2 \right) \ . \tag{3.13}$$

The a.s. limit for (non-random) integrated density of states $\mathcal{N}_\lambda(E)$ exists by the *Birkhoff ergodic theorem* [2, 22] and the uniform convergence of the s - sum ensures the permutation of expectation with respect the σ_λ - distribution (3.9) and the sum. Thus, we obtain:

$$\mathcal{N}_\lambda(E) \;=\; \lambda^2 \sum_{s=1}^{\infty} \int_{cs/\sqrt{E}}^{\infty} dL_i e^{-\lambda L_i} = \lambda \sum_{s=1}^{\infty} e^{-cs\lambda/\sqrt{E}} \;, \tag{3.14}$$

which yields the explicit formula (3.11). □

3.2 Formula (3.11) allows us to recover for all energies $E > 0$ the one-dimensional integrated density of states for the free operator t, i.e. the case when density of impurities $\lambda = 0$, cf. Remark 2.1 and (2.7):

$$\lim_{\lambda \downarrow 0} \mathcal{N}_\lambda(E) = \mathcal{N}_{\lambda=0}(E) = \frac{\sqrt{2}}{\pi}\sqrt{E} \;. \tag{3.15}$$

Notice that for the *Lebesgue-derivative* $n_\lambda(E) := d\mathcal{N}_\lambda(E)/dE$, i.e. for the *density of states* [14] Sect.4, this limit is not uniform in E in the vicinity of the spectrum edge $E = 0$. This confirms the argument, previously presented in [13], that the Bose-Einstein condensation in such random media can not be viewed as a perturbation of the free case, since this phenomenon is tightly related to the behaviour of $n_\lambda(E)$ near the edge [1, 13].

On the other hand, for $\lambda > 0$ and for E close to the edge of the spectrum, the integrated density of states (3.11) exhibits the *Lifshitz' tail* behaviour:

$$\mathcal{N}_\lambda(E) = \lambda e^{-c\lambda/\sqrt{E}} \left\{1 - O(e^{-2c\lambda/\sqrt{E}})\right\} \;,\; E \downarrow 0 \;, \tag{3.16}$$

see Remark 2.1. In this case $\lim_{E\downarrow 0} n_\lambda(E) = 0$.

It is known [11, 16] that behaviour (3.16) near the edge remains valid even if the parameter $a > 0$ in (2.6) is *finite*. Notice that this parameter does not appear in the leading term of the asymptotics (3.16). This can be explained by the fact that particle with small energy "sees" a point impurity potential with relative amplitude $a/E \gg 1$. Therefore, in spite of its singular nature the Luttinger-Sy Hamiltonian seems to be a good approximation for studying the BEC in Poisson random systems with non-singular repulsive impurities.

4 Thermodynamics and Bose-Einstein Condensation

The second quantization of the one-particle Luttinger-Sy Hamiltonian (3.4) in the boson Fock space gives the one-dimensional PBG embedded into a random potential created by Poisson repulsive impurities (2.6) with $a = +\infty$. The latter implies that bosons are distributed over independent intervals ("boxes") $\left\{L_j^\omega\right\}_{j,\omega}$.

4.1 We study the boson Luttinger-Sy model in the grand canonical ensemble, defined by the inverse temperature $\beta > 0$ and the chemical potential μ. Since the model corresponds to independent "boxes" $\left\{L_j^\omega\right\}_{j,\omega}$, the grand partition function of the PBG in $\Lambda = [-L/2, L/2]$ is the product of partition functions calculated in individual "boxes" :

$$\Xi_L^\omega(\beta,\mu) = \prod_{j=1}^{n} \Xi_{L_j}^\omega(\beta,\mu) = \prod_{j=1}^{n}\prod_{s=1}^{\infty}\left(1 - e^{-\beta(E_{s_j}(L_j^\omega)-\mu)}\right)^{-1} \;,$$

see (3.4). This gives for the grand canonical pressure

$$p_L^\omega(\beta,\mu) = -\frac{1}{\beta L}\sum_{j=1}^{n}\sum_{s=1}^{\infty}\ln\left(1-e^{-\beta(E_{s_j}(L_j^\omega)-\mu)}\right) \ . \tag{4.1}$$

To ensure the convergence in (4.1) we have to bound chemical potential from above: $\mu < \inf_{s_j,\omega} E_{s_j}(L_j^\omega)$. By virtue of (2.15) we obtain in the thermodynamic limit:

$$a.s.\lim_{L\to\infty}\inf_{s_j,\omega} E_{s_j}(L_j^\omega) = 0 \ . \tag{4.2}$$

Lemma 4.1 *For $\mu < 0$ and $L \to \infty$ the pressure $p_L^\omega(\beta,\mu)$ converges almost surely to the non-random function*

$$p(\beta,\mu) = a.s.\lim_{L\to\infty} p_L^\omega(\beta,\mu) = -\frac{1}{\beta}\int_0^\infty dE\ n_\lambda(E)\ln\left(1-e^{-\beta(E-\mu)}\right) \ , \tag{4.3}$$

where the limiting density of states

$$n_\lambda(E) := \frac{\lambda^2 c}{2}\frac{e^{c\lambda/\sqrt{E}}}{E^{3/2}\left(e^{c\lambda/\sqrt{E}}-1\right)^2} \ , \tag{4.4}$$

and $c = \pi/\sqrt{2}$, cf. (3.11).

Proof: By definition of integrated density of states (3.12) we can represent the pressure in (4.1) as the Lebesgue-Stieltjes integral

$$p_L^\omega(\beta,\mu) = -\frac{1}{\beta}\int_0^\infty d\mathcal{N}_L^\omega(E)\ln\left(1-e^{-\beta(E-\mu)}\right) \ .$$

Then by virtue of (3.13) we obtain that the limit

$$\begin{aligned} p(\beta,\mu) &= a.s.\lim_{L\to\infty} p_L^\omega(\beta,\mu) \\ &= -\frac{\lambda^2}{\beta}\sum_{s=1}^{\infty}\int_0^\infty dL_i e^{-\lambda L_i}\ln\left(1-e^{-\beta((cs/L_i)^2-\mu)}\right) \end{aligned}$$

exists a.s. and, after change of variables and calculation of the sum, takes the form:

$$p(\beta,\mu) = -\frac{\lambda^2}{\beta}\int_0^\infty \frac{dk}{k^2}\frac{e^{-c\lambda/k}}{1-e^{-c\lambda/k}}\ln\left(1-e^{-\beta(k^2-\mu)}\right) \ .$$

Setting $k = \sqrt{E}$, we recover the relation (4.3) with density of states (4.4). □

Similarly we obtain the statement about the thermodynamic limit of the grand-canonical particle density.

Lemma 4.2 *For all $\mu < 0$ and $\beta > 0$, the grand-canonical particles density*

$$\rho_L^\omega(\beta,\mu) = \frac{1}{L}\sum_{j=1}^{n}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(E_{s_j}(L_j^\omega)-\mu)}-1} = \int_0^\infty d\mathcal{N}_L^\omega(E)\ \frac{1}{e^{\beta(E-\mu)}-1} \ , \tag{4.5}$$

converges a.s. to

$$\rho(\beta,\mu) = \int_0^\infty dE\ \frac{n_\lambda(E)}{e^{\beta(E-\mu)}-1} \ , \tag{4.6}$$

with density of states $n_\lambda(E)$ defined by (4.4).

Proof: By virtue of representation (4.5), the demonstration follows the same line of reasoning as we used above for the limiting pressure (4.3). □

Corollary 4.1 *By explicit formula (4.4) we obtain that for the Luttinger-Sy model, defined by the Hamiltonian (2.11), the critical density*

$$\rho_c(\beta) = \lim_{\mu \uparrow 0} \int_0^\infty dE \frac{n_\lambda(E)}{e^{\beta(E-\mu)} - 1} \tag{4.7}$$

is bounded.

4.2 It is known that for PBG the condition $\rho_c(\beta) < \infty$ implies the existence of (*generalized*) Bose condensation [1], when the particles density $\rho > \rho_c(\beta)$. To make it obvious in our case we have to study solutions $\mu_L^\omega(\beta, \rho)$ of the finite-volume equations, see (4.5):

$$\rho = \rho_L^\omega(\beta, \mu) \ , \ \omega \in \Omega \ . \tag{4.8}$$

In fact, the asymptotic behaviour of $\mu_L^\omega(\beta, \rho)$ studied for a general ergodic non-negative random potential in [13]. These results then can be applied to the Luttinger-Sy model and lead to the following proposition:

Proposition 4.1 *Let $\mu_L^\omega(\beta, \rho)$ be solution of the equation (4.8) for a given $\omega \in \Omega$. Then*
(a) *for $\rho < \rho_c(\beta)$ the limit*

$$a.s. \lim_{L \to \infty} \mu_L^\omega(\beta, \rho) = \mu(\beta, \rho) < 0 \ , \tag{4.9}$$

exists and is the unique root of equation defined by (4.6):

$$\rho = \rho(\beta, \mu) \ , \tag{4.10}$$

(b) *for $\rho \geq \rho_c(\beta)$ the limit*

$$a.s. \lim_{L \to \infty} \mu_L^\omega(\beta, \rho) = 0 \ , \tag{4.11}$$

For $\rho \geq \rho_c(\beta)$ the limit (4.11) implies that the density of condensed particles can be define in the usual (for *generalized* condensation) way:

$$\rho_0(\beta, \rho) := \lim_{\epsilon \downarrow 0} \left\{ a.s. \ \lim_{L \to \infty} \int_0^\epsilon \mathcal{N}_L^\omega(dE) \frac{1}{e^{\beta(E - \mu_L^\omega(\beta,\rho))} - 1} \right\} = \rho - \rho_c(\beta) \ , \tag{4.12}$$

see e.g. [1]. If $\rho < \rho_c(\beta)$, the limit is zero. Notice that this nonrandom limit is a consequence of the chemical potential asymptotics (Proposition 4.1) and of the uniform convergence of the particle density (4.5), see [13], Theorem 4.1.

5 Off-Diagonal Long-Range Order

In this section we study the problem the *two-point* correlation function [21, 24]. By definition, in the finite volume Λ and for any $\omega \in \Omega$, it has the form:

$$\begin{aligned} \rho_\omega^L(x, y; \beta, \mu) : &= \sum_{j=1}^n \sum_{s_j=1}^\infty \frac{1}{e^{\beta(E_{s_j}(L_j^\omega) - \mu)} - 1} \left(\overline{\Psi_{s_j,D,n}^{L,\omega}(x)}, \Psi_{s_j,D,n}^{L,\omega}(y) \right)_{\mathbb{R}^n} \\ &= \sum_{j=1}^n \sum_{s_j=1}^\infty \frac{1}{e^{\beta(E_{s_j}(L_j^\omega) - \mu)} - 1} \overline{\psi_{j,s}^\omega}(x) \ \psi_{j,s}^\omega(y) \ , \end{aligned} \tag{5.1}$$

where $(\cdot\,,\,\cdot)_{\mathbb{R}^n}$ denotes the scalar product in $\mathbb{R}^n$, see (3.4) and (3.5). Therefore (5.1) is the kernel of for the *one-body reduced density matrix*, see e.g. [15].

5.1 We know that this function is *not self-averaging* in the thermodynamic limit [16, 20]. To get a way out, we proposed in [13] to consider the *space-averaged* version of (5.1):

$$\tilde{\rho}^L_\omega(x,y;\beta,\mu) := \frac{1}{L}\int_{-L/2}^{L/2} dz\ \rho^L_\omega(\beta,\mu;x+z,y+z)\ . \tag{5.2}$$

The motivation was based on the fact that in the limit $\lambda \downarrow 0$, we recover from (5.2) the free case, see [13] and Section 5.2 below.

Theorem 5.1 *For the Luttinger-Sy model, the thermodynamic limit of the space-averaged two-point correlation function (5.2) for $\beta > 0$ and $\mu \leq 0$, is a.s. nonrandom and has the form:*

$$\tilde{\rho}(x,y;\beta,\mu) = \rho_0(\beta,\rho) + e^{-\lambda|x-y|}\int_0^\infty dE \frac{n_\lambda(E)}{e^{\beta(E-\mu)}-1}\cos(\sqrt{2E}(x-y))\ . \tag{5.3}$$

Here $n_\lambda(E)$ is defined by (4.4) and $\rho_0(\beta,\rho)$ the condensate density (4.12).

Proof: We consider first the case of negative chemical potential (4.9), i.e. $\rho_0(\beta,\rho) = 0$. Using explicit form of eigenfunctions (2.16, we obtain for the thermodynamic limit of (5.2):

$$\begin{aligned}\tilde{\rho}(x,y;\beta,\mu) = \lim_{L\to\infty}\frac{\lambda}{n}\sum_{j=1}^{n}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(E_{s_j}(L^\omega_j)-\mu)}-1}\times \\ \frac{2}{L_j}\int_0^L da \sin(k_{s_j}(L^\omega_j)(x+a-y^\omega_{j-1}))\sin(k_{s_j}(L^\omega_j)(y+a-y^\omega_{j-1}))\mathbb{I}_{I^\omega_j}(x+a)\mathbb{I}_{I^\omega_j}(y+a)\ ,\end{aligned} \tag{5.4}$$

with $k_{s_j}(L^\omega_j) := \sqrt{2E_{s_j}(L^\omega_j)}$. Let us put, for simplicity, $x > y$. Then the integration is reduced to $[y^\omega_j - y, y^\omega_j - x + L^\omega_j]$, such that $(x-y) \leq L^\omega_j$. Since $k_{s_j}(L^\omega_j)L^\omega_j = s\pi$, the integration over a yields

$$\begin{aligned}&\frac{2}{L_j}\int_0^L da \sin(k_{s_j}(L^\omega_j)(x+a-y^\omega_{j-1}))\sin(k_{s_j}(L^\omega_j)(y+a-y^\omega_{j-1}))\mathbb{I}_{I^\omega_j}(x+a)\mathbb{I}_{I^\omega_j}(y+a) = \\ &\cos(k_{s_j}(L^\omega_j)(x-y))\theta(L^\omega_j-(x-y))\left(1-\frac{x-y}{L^\omega_j}\right),\end{aligned}$$

with step function $\theta(z)$. Since by Proposition 3.1 random variables L^ω_j are independent in the limit $L \to \infty$, we apply to (5.4) the Birkhoff ergodic theorem and find the limit:

$$\begin{aligned}\tilde{\rho}(x,y;\beta,\mu) &= a.s.\ \lambda^2\sum_{s=1}^{\infty}\int_0^\infty dL_j e^{-\lambda L_j}\frac{1}{e^{\beta((cs/L_j)^2)-\mu)}-1}\times \\ &\cos(\sqrt{2}cs(x-y)/L_j)\theta(L_j-(x-y))\left(1-\frac{x-y}{L_j}\right)\ .\end{aligned}$$

with $c = \pi/\sqrt{2}$. If we put $q = cs/L_j$, then

$$\begin{aligned}\tilde{\rho}(x,y;\beta,\mu) &= \lambda^2\sum_{s=1}^{\infty}\int_0^\infty \frac{dq}{q^2}\ e^{-cs\lambda/q}\ \frac{1}{e^{\beta(q^2-\mu)}-1}\times \\ &\cos(\sqrt{2}q(x-y))\ \theta(s-q(x-y)/c)\ c\ \{s-q(x-y)/c\}\ .\end{aligned} \tag{5.5}$$

The sum over s yields

$$\sum_{s\geq s_{min}}^{\infty} e^{-cs\lambda/q}(s-q(x-y)/c) = e^{-\lambda(x-y)}\frac{e^{-c\lambda/q}}{(1-e^{-c\lambda/q})^2} \ , \tag{5.6}$$

where $s_{min} = [q(x-y)/c]$ denotes the *entire* part of $q(x-y)/c$. Then after change of variables, $q = \sqrt{E}$, we find by (4.4) and (5.5) for $\mu < 0$:

$$\tilde{\rho}(x,y;\beta,\mu) = \ e^{-\lambda|x-y|}\int_0^{\infty} dE \ \frac{n_\lambda(E)}{e^{\beta(E-\mu)}-1} \ \cos(\sqrt{2E}(x-y)) \ . \tag{5.7}$$

We put here $|x-y|$, since the proof for $x-y\leq 0$ is identical to that for $0\leq x-y$.

Now we shall study the case when the condensate exists. The finite-volume chemical potential $\mu_L^\omega(\beta,\rho)$ is a solution of equation (4.8). By Proposition 4.1(b) for $\rho > \rho_c(\beta)$ it implies (4.11), i.e. $\mu_L^\omega(\beta,\rho>\rho_c(\beta))$ converges a.s. to 0. To find the limit of space-averaged correlation function (5.2) for the sequence $\{\mu_L^\omega(\beta,\rho)\}_L$ we split (5.2) into two parts:

$$\begin{aligned}
\tilde{\rho}_L^\omega(x,y;\beta,\mu_L^\omega(\beta,\rho)) &= \frac{1}{L}\sum_{j=1}^{n}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(E_{s_j}(L_j^\omega)-\mu_L^\omega(\beta,\rho))}-1}\times \\
&\cos\left(\sqrt{E_{s_j}(L_j^\omega)}(x-y)\right)\theta(L_j^\omega-(x-y))\left(1-\frac{(x-y)}{L_j^\omega}\right)\theta(\delta-E_{s_j}(L_j^\omega)) \\
&+\frac{1}{L}\sum_{j=1}^{n}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(E_{s_j}(L_j^\omega)-\mu_L^\omega(\beta,\rho))}-1}\times \\
&\cos\left(\sqrt{E_{s_j}(L_j^\omega)}(x-y)\right)\theta(L_j^\omega-(x-y))\left(1-\frac{(x-y)}{L_j^\omega}\right)\theta(E_{s_j}(L_j^\omega)-\delta) \ ,
\end{aligned} \tag{5.8}$$

for some $\delta > 0$. Since in the second term of the right-hand side of (5.8) the eigenvalues $E_{s_j}(L_j^\omega)\geq\delta>0$, the limit (4.11) and uniform convergence of the sums (cf. Corollary 3.1 in [13]) yields

$$\begin{aligned}
\lim_{\delta\downarrow 0}\lim_{L\to\infty}\tilde{\rho}_L^\omega(x,y;\beta,\mu_L^\omega(\beta,\rho)) &= a.s.\ \lim_{\delta\downarrow 0}\lim_{L\to\infty}\frac{1}{L}\sum_{i=1}^{\nu}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(E_{s_j}(L_j^\omega)-\mu_L^\omega(\beta,\rho))}-1}\times \\
&\cos\left(\sqrt{E_{s_j}(L_j^\omega)}(x-y)\right)\theta(L_i^\omega-(x-y))\left(1-\frac{(x-y)}{L_i^\omega}\right)\theta(\delta-E_{s_j}(L_j^\omega)) \\
&+\tilde{\rho}(x,y;\beta,0) \ ,
\end{aligned} \tag{5.9}$$

where the last term is defined by (5.7).

To study the limit of the first term in the right-hand side of (5.9) we use the fact that the levels with energies $E_{s_j}(L_j^\omega)<\delta$ are, by definition, of order $O(\delta)$. By (2.15) these levels are defined in large "boxes" of lengths of order $O(\delta^{-1/2})$. Then for $E_{s_j}(L_j^\omega)<\delta$ and $|x-y|\ll\delta^{-1/2}$ with δ small enough, we obtain asymptotics

$$\cos\left(\sqrt{E_{s_j}(L_j^\omega)}(x-y)\right)\theta(L_i^\omega-(x-y))\left(1-\frac{(x-y)}{L_i^\omega}\right) = 1+O\left(\sqrt{\delta}\right) \ .$$

Therefore, by definition of *generalized* condensation (4.12) and by (5.9), we get

$$\tilde{\rho}_L^\omega(x,y;\beta,0) := a.s.\ \lim_{\delta\downarrow 0}\lim_{L\to\infty}\tilde{\rho}_L^\omega(x,y;\beta,\mu_L^\omega(\beta,\rho)) = \rho_0(\beta,\rho)+\tilde{\rho}(x,y;\beta,0) \ , \tag{5.10}$$

for $\rho > \rho_c(\beta)$. This finishes the proof of (5.3). $\square$

5.2 It is instructive to make a contact between the concept of the *space-averaged* one-body reduced density matrix (5.2) and the one for nonrandom free PBG.

Corollary 5.1 *When density of particles ρ exceeds the critical value $\rho_c(\beta)$ (4.7), the space-averaged one-body reduced density matrix of the Luttinger-Sy model, manifests Off-Diagonal Long-Range Order:*

$$ODLRO(\beta, \rho) := \lim_{|x-y|\to\infty} \tilde{\rho}(x, y; \beta, \mu(\beta, \rho)) = \rho_0(\beta, \rho) \ , \tag{5.11}$$

We see that similarly to the nonrandom case this limit is defined by the condensation density (4.12).

Remark 5.1 *Notice that for $\mu < 0$ (or $\rho < \rho_c(\beta)$) , the space-averaged reduced density matrix (5.3) remains consistent with the free nonrandom case. Indeed, by (4.4) we get for (5.7) the limit*

$$\lim_{\lambda\downarrow 0} \tilde{\rho}(x, y; \beta, \mu) = \rho(x, y; \beta, \mu) \ , \tag{5.12}$$

where

$$\rho(x, y; \beta, \mu) = \frac{1}{\pi} \int_0^\infty \frac{dE}{\sqrt{2E}} \ \frac{1}{e^{\beta(E-\mu)} - 1} \ \cos(\sqrt{2E}(x-y)) \tag{5.13}$$

coincides with two-point correlation function of the free PBG [13]. This equivalence is valid only when there is no condensation, since by (4.4) and (4.7) one has that $\lim_{\lambda\downarrow 0} \rho_c(\beta) = \infty$.

Let the single-impurity potential $u(x) \geq 0$, $x \in \mathbb{R}$, have compact support $[-\eta/2, \eta/2]$, see Section 2.2. Then one has

Proposition 5.1 [13] *Let $\tilde{\gamma} := 1 - e^{-\tilde{v}}$, with $\tilde{v} := \int_{\mathbb{R}} v(x)dx$. Then*

$$\tilde{\rho}(x, y; \beta, \mu) \leq \rho(x, y; \beta, \mu) \ e^{-\lambda\tilde{\gamma}(|x-y|-\eta)} \ , \tag{5.14}$$

$\mathbb{P}-a.s.$ for any $\mu < 0$.

Since for the Luttinger-Sy model the single-impurity potential is defined as a singular δ-point potential with infinite amplitude (Section 2.4), we get in (5.14) $\tilde{\gamma} = 1$ and $\eta = 0$.

To check this directly, put $E = w|x-y|^2$. Then for $\mu < 0$ we can represent two-point correlation function (5.7) as

$$\begin{aligned}
\tilde{\rho}(x, y; \beta, \mu) &= Re \ e^{-\lambda|x-y|} \sum_{s=1}^{\infty} e^{\beta\mu s} \int_0^\infty dE e^{-\beta E s} n_\lambda(E) e^{i\sqrt{2E}|x-y|} \\
&= e^{-\lambda|x-y|}|x-y|^2 \ Re \sum_{s=1}^{\infty} e^{\beta\mu s} \int_0^\infty dw \ n_\lambda(w|x-y|^2) \ e^{-|x-y|^2(\beta w s - i\sqrt{2w})} \ .
\end{aligned} \tag{5.15}$$

To calculate the asymptotics of $\tilde{\rho}(x, y; \beta, \mu)$, when $|x-y| \to \infty$, we estimate the last integral in (5.15) by the *saddle-point* method. Then (4.4) implies

$$\begin{aligned}
\tilde{\rho}(x, y; \beta, \mu) &= e^{-\lambda|x-y|} \left\{ \sum_{s=1}^{\infty} e^{\beta\mu s} \frac{e^{-|x-y|^2/4\beta s}}{(2\pi\beta s)^{1/2}} + e^{-\sqrt{2|\mu|}|x-y|} O(|x-y|^{-1}) \right\} \\
&= e^{-\lambda|x-y|} \ \rho(x, y; \beta, \mu) \ + \ e^{-(\sqrt{2|\mu|}+\lambda)|x-y|} \ O(|x-y|^{-1}) \ ,
\end{aligned}$$

for $|x-y| \to \infty$, with the free PBG two-point correlation function $\rho(x, y; \beta, \mu)$ defined by (5.13). This confirms the statement (5.14) for the case of the Luttinger-Sy model.

6 Comments and discussion

6.1 Critical density

We start by remark concerning modifications of the Luttinger-Sy model properties, and in particular of the value of the critical density, when one passes from infinite to a *finite* amplitude $a < \infty$ of the δ-potential (2.6).

Recall that operators $\{h_a^\omega\}_{a\geq 0}$ correspond to a monotonically increasing family of quadratic forms with $h_{a=+\infty}^\omega = h_D^\omega$, see (2.8). Then by definition of the integrated density of states (3.10), (3.11) and by the *mini-max* principle for h_D^ω and h_a^ω, one gets

$$\mathcal{N}_\lambda(E) \equiv \mathcal{N}_{\lambda,a=+\infty}(E) < \mathcal{N}_{\lambda,a}(E) \leq \mathcal{N}_{\lambda,a=0}(E) = \mathcal{N}_{\lambda=0,a}(E) = \frac{\sqrt{E}}{c} \ . \tag{6.1}$$

Notice that the integrated density of states for the free case $a = 0$ coincides with that for the zero impurity density (3.15), $\lambda = 0$. From (6.1) one gets the corresponding inequalities for critical densities (4.7) indicating the enhancement of the BEC, that was remarked already by Kac and Luttiger [8, 9].

More refined arguments (see e.g. [20], Ch.III, 6B) give for $E < \pi^2 a^2/32$, the estimate, cf. (3.14):

$$\mathcal{N}_\lambda(E) < \mathcal{N}_{\lambda,a}(E) < \lambda \sum_{s=1}^{\infty} e^{-s\lambda(c/\sqrt{E}-4/a)} =: \mathcal{N}_{\lambda,a}^*(E) \ . \tag{6.2}$$

Then, definition of the critical density (4.7) and (6.1) yield

$$\begin{aligned}
\rho_c(\beta,\lambda) : &= \lim_{\mu\uparrow 0}\int_0^\infty \frac{d\mathcal{N}_\lambda(E)}{e^{\beta(E-\mu)}-1} = \int_0^\infty dE\ \mathcal{N}_\lambda(E)\ \frac{\beta e^{\beta E}}{(e^{\beta E}-1)^2} \\
&\leq \int_0^\infty dE\ \mathcal{N}_{\lambda,a}(E)\ \frac{\beta e^{\beta E}}{(e^{\beta E}-1)^2} =: \rho_c(\beta,\lambda,a) \ ,
\end{aligned} \tag{6.3}$$

with obvious limits: $\lim_{a\to 0}\rho_c(\beta,\lambda,a) = \infty$ and $\lim_{\lambda\to 0}\rho_c(\beta,\lambda,a) = \infty$, by (6.1).

The estimate (6.2) yields the upper bound on $\rho_c(\beta,\lambda,a)$ for small a. Setting $\tilde{E}(a) := (\pi a/8)^2$ by (6.1)-(6.3) we get that

$$\rho_c(\beta,\lambda,a) \leq \int_0^{\tilde{E}(a)} dE\ \mathcal{N}_{\lambda,a}^*(E)\ \frac{\beta e^{\beta E}}{(e^{\beta E}-1)^2} + \int_{\tilde{E}(a)}^\infty dE\ \frac{\sqrt{E}}{c}\ \frac{\beta e^{\beta E}}{(e^{\beta E}-1)^2} =: I(\beta,\lambda,a) \ . \tag{6.4}$$

Then for fixed $\lambda > 0$ and small $a > 0$ we obtain the estimate:

$$\rho_c(\beta,\lambda,a) \leq I(\beta,\lambda,a) \leq \frac{1}{\beta\lambda}\left(\frac{8}{\pi}\right)^2 \frac{1}{4e(\sqrt{2}-1)} + \frac{1}{a}\ \frac{16\sqrt{2}}{\beta\pi^2} \ .$$

Similarly, for $a > 0$ and small $\lambda > 0$ we obtain by (6.4) that

$$\rho_c(\beta,\lambda,a) \leq \frac{1}{\lambda\beta}\int_0^{\tilde{E}(a)} \frac{dx}{x^2}\frac{e^{-c/\sqrt{x}}}{1-e^{-c/\sqrt{x}}} + \int_{\tilde{E}(a)}^\infty dE\ \frac{\sqrt{E}}{c}\ \frac{\beta e^{\beta E}}{(e^{\beta E}-1)^2} \ .$$

Notice that the *bounded* critical density $\rho_c(\beta,\lambda,a)$ for $\lambda > 0$ and $a > 0$ is the key criterium of existence of BEC in the one-dimensional system with integrated density of states $\mathcal{N}_{\lambda,a}(E)$, cf. (4.12). On the other hand the Bogoliubov-Hohenberg theorem says that there is no BEC in *translation invariant* boson systems if dimension is less than $d = 2$, see e.g. [3, 4]. Therefore, the BEC in the Luttinger-Sy model is of a different nature than the case without random impurity potential.

In fact, the randomness of the impurity potential is not indispensable for BEC in one-dimensional perfect Bose-gas. To this end we construct *nonrandom hierarchical models* with impurity potential which manifest the BEC via mechanism similar to that in the Luttinger-Sy model.

6.2 Hierarchical model for BEC in one-dimensional nonrandom intervals

We present here a nonrandom *hierarchical* one-dimensional system, which manifests BEC and in a certain sense mimics the Luttinger-Sy model.

Type I BEC. Let $\Lambda := (0, L)$ be a segment separated into n *impenetrable* intervals of lengths L_j, $j = 1, ..., n$ such that $\lambda = n/L < \infty$. For simplicity we take the hierarchy when all intervals, except the first (*largest*) one, are identical:

$$L_1 = \frac{\ln(\lambda L)}{\lambda} \quad \text{and} \quad L_{j\neq 1} = \tilde{L}_n = \frac{L - L_1}{n-1} \ . \tag{6.5}$$

Then one gets

$$\lim_{L\to\infty} L_1 = +\infty \quad \text{and} \quad \lim_{L\to\infty} \tilde{L}_n = \frac{1}{\lambda} \ . \tag{6.6}$$

This non-random system presents an obvious analogue of the Luttinger-Sy model. Here again, the quantum states are defined in independent intervals and they have energies

$$E_{j,s} = \frac{c^2 s^2}{L_j^2} \ , \ j = 1, ..., n \ , \ s = 1, 2, ... \ , \tag{6.7}$$

with $c^2 = \pi^2/2$. The spectrum of the corresponding Schrödinger operator is discrete and bounded below by zero, cf. (2.15), (2.17). Then the chemical potential is $\mu < 0$ and the PBG particle density in Λ has the same expression as in (4.5)

$$\rho_L(\beta, \mu) = \frac{1}{L} \sum_{j=1}^{n} \sum_{s=1}^{\infty} \frac{1}{e^{\beta(E_{j,s}-\mu)} - 1} \ , \ \beta > 0 \ , \ \mu \leq 0 \ .$$

By virtue of the hierarchical structure of intervals we can separate the expression for density into two parts:

$$\rho_L(\beta, \mu) = \frac{1}{L} \sum_{s=1}^{\infty} \frac{1}{e^{\beta(c^2 s^2/L_1^2 - \mu)} - 1} + \frac{n-1}{L} \sum_{s=1}^{\infty} \frac{1}{e^{\beta(c^2 s^2/\tilde{L}_n^2 - \mu)} - 1} \ . \tag{6.8}$$

Since $L_1 = O(\ln(\lambda L))$, the first sum in (6.8) converges, when $L \to \infty$, to zero for all $\mu \leq 0$, i.e. we obtain

$$\rho(\beta, \mu) = \lim_{L\to\infty} \rho_L(\beta, \mu) = \lambda \sum_{s=1}^{\infty} \frac{1}{e^{\beta((cs/\lambda)^2 - \mu)} - 1} \ . \tag{6.9}$$

As a consequence, the critical density for this system is finite:

$$\rho_c(\beta) := \sup_{\mu\leq 0} \rho(\beta, \mu) = \rho(\beta, 0) = \lambda \sum_{s=1}^{\infty} \frac{1}{e^{\beta(cs/\lambda)^2} - 1} < \infty \ , \tag{6.10}$$

and we have BEC condensation, when $\rho > \rho_c(\beta)$.

This condensation is of *type I*, since the difference between the *ground-state* energy and the energy of the *first excited* state (which are both localized in the biggest interval L_1) is of the order $O(L_1^{-2}) = O((\ln(\lambda L))^{-2})$, see e.g. [1], or [25]. In this case the solution $\mu_L(\beta, \rho)$ of the equation

$$\begin{aligned}\rho &= \frac{1}{L}\frac{1}{e^{\beta(c^2/L_1^2-\mu)}-1} + \frac{1}{L}\sum_{s>1}\frac{1}{e^{\beta((cs/L_1)^2-\mu)}-1} \\ &+ \frac{\nu-1}{L}\sum_{s=1}^{\infty}\frac{1}{e^{\beta((cs/\tilde{L}_n)^2-\mu)}-1}\ ,\end{aligned} \tag{6.11}$$

for $\rho > \rho_c(\beta)$ and large L has asymptotics

$$\mu_L(\beta, \rho) = E_{1,1} - \frac{1}{\beta\rho_0(\beta, \rho)L} + O(1/L^2)\ , \tag{6.12}$$

see (6.7). Inserting (6.12) into (6.11) we obtain in the limit

$$\begin{aligned}\rho &= \lim_{L\to\infty}\frac{1}{L}\frac{1}{e^{\beta(c^2/L_1^2-\mu_L(\beta,\rho))}-1} + \lambda\sum_{s=1}^{\infty}\frac{1}{e^{\beta(cs/\lambda)^2}-1} \\ &= \rho_0(\beta, \rho) + \rho_c(\beta)\ ,\end{aligned}$$

where by (6.12) the *condensate density* is

$$\rho_0(\beta, \rho) = \lim_{L\to\infty}\frac{1}{L}\ \frac{1}{e^{\beta(c^2/L_1^2-\mu_L(\beta,\rho))}-1}\ .$$

Therefore, this one-dimensional hierarchical model shows a *type I* BEC localized in the (*logarithmically*) large, but not macroscopic, domain corresponding to the ground-state wave function. Generalizations to another hierarchy of intervals with one largest interval trapping BEC are obvious.

For example, it is easy to generalize the above observation to the *type I* BEC localized in a *finite* number of M identical (*logarithmically*) large intervals:

$$L_1 = \ldots = L_M = \frac{\ln(\lambda L)}{\lambda} \quad \text{and} \quad L_j = \tilde{L}_n = \frac{L - ML_1}{n - M}\ ,\ M < j \leq n\ , \tag{6.13}$$

cf. (6.5). Then similar to the case $M = 1$ (6.8) one gets

$$\rho_L(\beta, \mu) = \frac{1}{L}\sum_{j=1}^{M}\ \sum_{s=1}^{\infty}\frac{1}{e^{\beta(c^2s^2/L_j^2-\mu)}-1} + \frac{n-M}{L}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(c^2s^2/\tilde{L}_n^2-\mu)}-1}\ . \tag{6.14}$$

which implies trough verbatim that for $M > 1$ the critical density (6.10) rests the same. If $\rho > \rho_c(\beta)$, the equation $\rho = \rho_L(\beta, \mu)$ (6.14) yields for asymptotics of the solution $\mu_L(\beta, \rho)$ an expression similar to (6.12) for $M = 1$:

$$\mu_L(\beta, \rho) = E_{j,1} - \frac{M}{\beta\rho_0(\beta, \rho)L} + O(1/L^2)\ ,\ 1 \leq j \leq M\ . \tag{6.15}$$

Here $E_{1,1} = \ldots = E_{M,1}$ by (6.7) and (6.13). Then taking into account (6.7) and (6.13), (6.15) we obtain by (6.14)

$$\begin{aligned} \rho &= \lim_{L\to\infty} \frac{1}{L}\sum_{j=1}^{M} \frac{1}{e^{\beta(c^2/L_j^2 - \mu_L(\beta,\rho))} - 1} + \lambda \sum_{s=1}^{\infty} \frac{1}{e^{\beta(cs/\lambda)^2} - 1} \\ &= \rho_0(\beta,\rho) + \rho_c(\beta) \ , \end{aligned}$$

where the condensate density is *equally* shared among the first M intervals:

$$\rho_0(\beta,\rho) = \lim_{L\to\infty} \frac{M}{L} \frac{1}{e^{\beta(c^2/L_1^2 - \mu_L(\beta,\rho))} - 1} \ .$$

Type II BEC. To obtain the *type II* BEC in *one* interval we take, instead of (6.5):

$$L_1 := \sqrt{L/\lambda} \quad \text{and} \quad L_{j\neq 1} = \tilde{L}_n = \frac{L - L_1}{n-1} \ . \tag{6.16}$$

Then (6.8) gets the form

$$\rho_L(\beta,\mu) = \frac{1}{L}\sum_{s=1}^{\infty} \frac{1}{e^{\beta(\lambda c^2 s^2/L - \mu)} - 1} + \frac{n-1}{L}\sum_{s=1}^{\infty} \frac{1}{e^{\beta(c^2 s^2/\tilde{L}_n^2 - \mu)} - 1} \ . \tag{6.17}$$

Since for $\mu \leq 0$ the first sum in (6.17) converges to zero, when $L \to \infty$, we obtain for $\rho(\beta,\mu) = \lim_{L\to\infty} \rho_L(\beta,\mu)$ and $\rho_c(\beta)$ the same expressions as in (6.9) and (6.10).

Now, if $\rho > \rho_c(\beta)$, the solution $\mu_L(\beta,\rho)$ of equation $\rho = \rho_L(\beta,\mu)$ has asymptotics defined by (6.17):

$$\begin{aligned} \rho &= \lim_{L\to\infty} \frac{1}{L}\sum_{s=1}^{\infty} \frac{1}{e^{\beta(\lambda c^2 s^2/L - \mu_L(\beta,\rho))} - 1} + \lambda \sum_{s=1}^{\infty} \frac{1}{e^{\beta(cs/\lambda)^2} - 1} \\ &= \rho_0(\beta,\rho) + \rho_c(\beta) \ . \end{aligned} \tag{6.18}$$

As in (6.12) this implies

$$\mu_L(\beta,\rho) = E_{1,1} - \frac{A(\beta,\rho)}{\beta L} + O(1/L^2) \ , \tag{6.19}$$

see (6.7), where by (6.18) the coefficient $A(\beta,\rho) \geq 0$ satisfies the equation

$$\rho = \sum_{s=1}^{\infty} \frac{1}{\beta\lambda c^2(s^2-1) + A} + \rho_c(\beta) \ . \tag{6.20}$$

Hence, for $\rho > \rho_c(\beta)$ the BEC

$$\rho_0(\beta,\rho) = \sum_{s=1}^{\infty} \frac{1}{\beta\lambda c^2(s^2-1) + A(\beta,\rho)}$$

is splitted between *infinitely* many states in the largest interval L_1, i.e. this is condensation of the *type II*, [1].

Type III BEC. Now we show that (unusual) *spatially* fragmented *type III* BEC is possible in our hierarchical model. To split BEC between *infinitely* many states in *different* intervals, let volume Λ be occupied by $[\ln(n+1)]$ identical (*logarithmically*) large intervals:

$$L_j = \frac{\ln(\lambda L)}{\lambda} \quad , \quad 1 \leq j \leq [\ln(n+1)] =: M_n \quad \text{and} \quad L_{j>M_n} = \tilde{L}_n := \frac{L - L_1 M_n}{n - M_n} \ , \tag{6.21}$$

for $M_n < j \leq n$, of small intervals, cf. (6.13). Then (similar to (6.14)) we get for the particle density

$$\rho_L(\beta,\mu) = \frac{1}{L}\sum_{j=1}^{M_n}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(c^2s^2/L_j^2-\mu)}-1} + \frac{n-M_n}{L}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(c^2s^2/\tilde{L}_n^2-\mu)}-1} \ . \tag{6.22}$$

Since by (6.21) we have $\lim_{L\to\infty}\tilde{L}_n = \lim_{L\to\infty}(n-M_n)/L = \lambda$, then the critical density (6.10) rests the same. If $\rho > \rho_c(\beta)$, the equation $\rho = \rho_L(\beta,\mu)$ (6.22) yields for asymptotics of the solution $\mu_L(\beta,\rho)$:

$$\mu_L(\beta,\rho) = E_{j,1} - \frac{M_n}{\beta\rho_0(\beta,\rho)L} + O(1/L^2) \ , \ 1 \leq j \leq M_n \ . \tag{6.23}$$

Then we obtain for the particle density in large intervals

$$\begin{aligned}
\lim_{L\to\infty}\frac{1}{L}\sum_{j=1}^{M_n}\sum_{s=1}^{\infty}\frac{1}{e^{\beta(c^2s^2/L_j^2-\mu_L(\beta,\rho))}-1} &= \lim_{L\to\infty}\frac{M_n}{L}\frac{1}{e^{M_n/(\rho_0(\beta,\rho)L)-O(1/L^2)}-1} \\
+\lim_{L\to\infty}\frac{M_n}{L}\sum_{s=2}^{\infty}\frac{1}{e^{\beta((\lambda cs/\ln n)^2-\mu_L(\beta,\rho))}-1} &= \rho_0(\beta,\rho) \ .
\end{aligned} \tag{6.24}$$

The limit (6.24) and (6.22), (6.23) imply that the condensate density $\rho_0(\beta,\rho) = \rho - \rho_c(\beta)$ is splitted between ground states of *infinitely* many *logarithmic* intervals in such a way that the condensate density in each interval is *zero*. This is a *spatially* fragmented *type III* BEC, which is different from the that corresponding to spread out over infinite number of states discussed, e.g., in [1, 25].

6.3 Statistics of large Poisson intervals

By virtue of **6.2** to discriminate between possible types of BEC in the Luttinger-Sy model one has to study statistics of the size $\{L_j^\omega\}_j$ of intervals induced by Poisson distributed point impurities, see Proposition 3.1.

In fact, the first attempt to elucidated this question is already contained in [18]. They gave some arguments in favour of that for large finite Λ the *largest* interval I_1^ω has a *typical* length of the *logarithmic* order:

$$L_1^\omega \sim \lambda^{-1}\ln(\lambda L) \ , \ L \to \infty \ . \tag{6.25}$$

Moreover, along the same line of reasoning they conclude that all other intervals $I_{j>1}^\omega$ are typically much smaller than I_1^ω. Then neglecting the fluctuations the length of L_1^ω they conclude that BEC has to follow scenario we described in **6.2** as *type I* BEC, cf. (6.5) and (6.25).

To check these arguments and to bolster them by some rigorous reasonings we use Proposition 3.1. First we note that the average length of the Poisson intervals is

$$\mathbb{E}_{\sigma_\lambda}(L_{j_s}^\omega) = \lambda\int_0^\infty dL\, L\, e^{-\lambda L} = \lambda^{-1} \quad , \tag{6.26}$$

i.e., the total average length of any sample of intervals $\{I_j^\omega\}_{j=1}^k$ is k/λ. We are interested into density of the joint probability distributions generated by the events: $\{\omega \in \Omega : L_{j_1}^\omega \geq L_{j_2}^\omega \geq \ldots \geq L_{j_k}^\omega\}$. They have evidently the form

$$d\sigma_{\lambda,k}^{>}(L_{j_1},\ldots,L_{j_k}) := k!\ \theta(L_{j_1}-L_{j_2})\,\theta(L_{j_2}-L_{j_3})...\theta(L_{j_{k-1}}-L_{j_k})\ d\sigma_{\lambda,k}(L_{j_1},\ldots,L_{j_k})\ . \tag{6.27}$$

Then one gets for the joint probability density of the two *largest* intervals:

$$d\sigma_{\lambda,k}^{>}(L_{j_1},L_{j_2})/dL_{j_1}dL_{j_2} = k(k-1)\,\lambda^2\, e^{-\lambda L_{j_1}} e^{-\lambda L_{j_2}} \left(1-e^{-\lambda L_{j_2}}\right)^{k-2} \theta(L_{j_1}-L_{j_2})\ . \tag{6.28}$$

Now, let $A(s,t)$ be defined for non-negative integers s and t by

$$A(s,t) := \int_0^\infty dx\, \ln^s(x) x^t e^{-x}\ . \tag{6.29}$$

For $s \geq 1$ and $t \geq 1$ this function verifies the following identities:

$$A(s,t) = tA(s,t-1) + sA(s-1,t-1)\ ,\ A(1,t) = tA(1,t-1) + \Gamma(t)\ , \tag{6.30}$$

where $\Gamma(t)$ stands for the Gamma-function. Then by (6.28) and (6.29), (6.30) we obtain for expectations of the two largest intervals:

$$\mathbb{E}_{\sigma_{\lambda,k}^{>}}(L_{j_1}^\omega) = \frac{A(1,k)}{k!\ \lambda} - \frac{A(1,0)}{\lambda} = \frac{1}{\lambda}\sum_{s=1}^k \frac{1}{s}\ , \tag{6.31}$$

$$\mathbb{E}_{\sigma_{\lambda,k}^{>}}(L_{j_2}^\omega) = \frac{A(1,k)}{k!\ \lambda} - \frac{A(1,1)}{\lambda} = \frac{1}{\lambda}\sum_{s=2}^k \frac{1}{s}\ . \tag{6.32}$$

By virtue of (6.31) and (6.32) the mean difference $\mathbb{E}_{\sigma_{\lambda,k}^{>}}(L_{j_1}^\omega - L_{j_2}^\omega) = 1/\lambda$ is *independent* of the number k of intervals in the sample, whereas they have, for large k, the *logarithmic* size (cf. (6.25)):

$$\mathbb{E}_{\sigma_{\lambda,k}^{>}}(L_{j\,1,2}^\omega) = \frac{1}{\lambda}\ln(k) + \frac{1}{\lambda}P_{1,2} + O(1/k)\ , \tag{6.33}$$

with respect the total average sample length k/λ, here $P_1 = \mathbf{C} := 0,577\ldots$, is the *Euler constant*, and $P_2 = \mathbf{C} - 1$. Using (6.27) and (6.29), (6.30) we find that the variance of the difference between two largest intervals in the sample is also *k-independent* and has the form:

$$Var_{\sigma_{\lambda,k}^{>}}(L_{j_1}^\omega - L_{j_2}^\omega) = \frac{1}{\lambda^2}\ . \tag{6.34}$$

Moreover, by the joint probability distribution (6.28) we obtain for any $\delta > 0$ that probability

$$\mathbb{P}\{\omega : L_{j_1}^\omega - L_{j_2}^\omega > \delta\} = e^{-\lambda\delta} \tag{6.35}$$

of the events $A_k(\delta) = \{\omega : L_{j_1}^\omega - L_{j_2}^\omega > \delta\}$, is independent of k for increasing sequence of samples $\{I_j^\omega\}_{j=1}^k$, when $k \to \infty$.

By **6.2** and (6.33), (6.34) we see that the *type II* or *III* BEC are impossible in the one largest *logarithmic* "box", since that total average length of the sample is k/λ. To exclude the *type I,II,III* condensations via a *space* fragmentation between, e.g., two "boxes", we have to estimate probability of events corresponding to the state-energy spacings between two largest intervals.

By **6.2** (see (6.15), (6.19), (6.23) and [13]) this spacing should be larger than *inverse* of the total sample length, which is $(k/\lambda)^{-1}$. To this end it is sufficient to estimate the probability of the event $S_k(a>0,\gamma>0)$ corresponding the spacing between ground states:

$$\mathbb{P}\{S_k(a,\gamma)\} := \mathbb{P}\{\omega : E_{s=1}(L^{\omega}_{j_2}(k)) - E_{s=1}(L^{\omega}_{j_1}(k)) > \frac{a}{k^{1-\gamma}}\} \ . \tag{6.36}$$

Here we denote the energies in the sample $\left\{I^{\omega}_j\right\}_{j=1}^{k}$ by

$$E_s(L^{\omega}_{j_r}(k)) = \frac{c^2 s^2}{(L^{\omega}_{j_r}(k))^2} \ , \ r=1,...,k \ , \ s=1,2,... \ . \tag{6.37}$$

Notice that by (6.35) we obtain that there is a kind "repulsion" between energy levels in different intervals. Indeed,

$$\begin{aligned} &\mathbb{P}\{S_k(a,\gamma)\} \geq \mathbb{P}\{\omega : L^{\omega}_{j_1}(k) - L^{\omega}_{j_2}(k) > \frac{a}{2c^2 k^{1-\gamma}} L^{\omega}_{j_1}(k)(L^{\omega}_{j_2}(k))^2\} = \\ &\int_0^{\infty}\int_0^{\infty} d\sigma^{>}_{\lambda,k}(x,y)\theta(x-y-\frac{a}{2c^2k^{1-\gamma}}xy^2) =: p_k(a,\gamma) \ , \end{aligned} \tag{6.38}$$

where $\lim_{k\to\infty} p_k(a, 0<\gamma<1) = 1$ by explicit calculations in (6.38). The same argument is valid for other than ground states as well as for intervals $\left\{I^{\omega}_{j_r}\right\}_{r\geq 2}^{k}$ instead of $I^{\omega}_{j_2}$. Therefore, in this limit with the probability 1 the spacing is too large for fragmentation of condensate between the largest and other intervals.

6.4 The Kac-Luttinger conjecture

The above arguments prove the Kac-Luttinger *conjecture* in the case of the one-dimensional random Poisson potential of point impurities: for PBG the BEC is of *type I* and it is *localized* in one "largest box".

To make this statement more precise recall that BEC exists only in the thermodynamic limit, which we construct as *increasing family* of samples of intervals $\left\{I^{\omega}_j\right\}_{j=1}^{k}$ induced by the point impurities on $\mathbb{R}$. Since these random variables are independent, we can choose increasing sequence of independent samples with one largest interval and with the property that

$$\lim_{k\to\infty} \sum_{r=1}^{k} \mathbb{P}\{S_k(a,\gamma)\} = \infty \ , \tag{6.39}$$

see (6.38). Then by the Borel-Cantelli lemma [22]

$$\mathbb{P}\ \{\overline{\lim}\ S_k(a,\gamma)\} = 1 \ , \tag{6.40}$$

where the event

$$\overline{\lim}\ S_k(a,\gamma) := \bigcap_{k=1}^{\infty}\bigcup_{l=k}^{\infty} S_l(a,\gamma)$$

means that infinitely many events $\{S_k(a,\gamma)\}_{k\geq 1}$ take place. Together with **6.3** the statement (6.40) mean that with probability 1 in the thermodynamic limit $\mathbb{R}$ the BEC is localized in a single "largest box", and this condensation is of the *type* I.

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References

[1] van den Berg M., Lewis J.T. and Pulé J.V., A general theory of Bose-Einstein condensation, *Helv. Phys. Acta.* **59**, 1271-1288 (1986)

[2] Billingsley P. *Convergence of probability measures* (Wiley, N.Y. 1968).

[3] Bogoliubov N.N. *Lectures on Quantum Statistics: Quasi-Averagies, vol. 2.* (Gordon and Breach, N.Y. 1970).

[4] Bouziane M. and Martin Ph.A. Bogoliubov inequality for unbounded operators and the Bose gas, *J.Math.Phys.* **17**, 1848-1851 (1976)

[5] Dorlas T.C., Pastur L.A. and Zagrebnov V.A. Condensation in a disordered infinite-range hopping Bose-Hubbard model, *arXiv:math-ph/0603047 v1, 18 Mar 2006.*

[6] Fisher M. P. A., Weichman P. B., Grinstein G. and Fisher D. S. Boson localization and the superfluid-insulator transition, *Phys. Rev.* B**40** 456-470 (1989).

[7] Frish H.L. and Lloyd S.P. Electron levels in a one-dimensional random lattice, *Phys.Rev.* **120**, 1179-1194 (1960)

[8] Kac M. and Luttinger J.M. Bose-Einstein Condensation in the presence of impurities, *J.Math.Phys.* **14**, 1626-1628 (1973).

[9] Kac M. and Luttinger J.M. Bose-Einstein Condensation in the presence of impurities II, *J.Math.Phys.* **15**, 183-186 (1974).

[10] Kobayashi M. and Tsubota M. Bose-Einstein condensation and superfluidity of a dilute Bose gas in a random potential, *Phys. Rev.*B **66**: 174516-1–7 (2002).

[11] Kotani S. On asymptotic behaviour of the spectra of one-dimensional Hamiltonian with certain random coefficient, *Publ. Res. Inst. Math. Sci.* **12**, 477-492 (1976).

[12] Lenoble O. *Fonctions de corrélation de systèmes quantiques aléatoires.* PhD, Université de la Méditerranée, 2003.

[13] Lenoble O., Pastur L.A. and Zagrebnov V.A. Bose-Einstein condensation in random potentials, *Comptes-Rendus de L'Académie des Sciences (Paris), Physique* **5**, 129-142 (2004)

[14] Leschke H., Müller P. and Warzel S., A survay of rigorous results on random Schrödinger operators for amorphous solids, *Markov Processes and Related Fields* **9**, 729-766 (2003)

[15] Lewis J.T. and Pulé J.V. The equilibrium states of the free boson gas, *Commun.Math.Phys.* **36**, 1-18 (1974)

[16] Lifshitz I.M., Gredeskul S.A. and Pastur L.A. *Introduction to the theory of disordered systems* (Wiley-Interscience, N.Y. 1989).

[17] Luttinger J.M. and Sy H.K. Low-Lying Energy Spectrum of a One-Dimensional Disordered System, *Phys. Rev.* A**7**, 701-712 (1973).

[18] Luttinger J.M. and Sy H.K. Bose-Einstein Condensation in a One-Dimensional Model with Random Impurities, *Phys. Rev.* A**7**, 712-720 (1973).

[19] Pastur L.A. Spectral properties of disordered systems in one-body approximation, *Comm. Math. Phys.* **75**, 179-196 (1980).

[20] Pastur L.A. and Figotin A. *Spectra of Random and Almost-Periodic Operators* (Springer-Verlag, Berlin 1992).

[21] Penrose O. and Onsager L. Bose-Einstein condensation and liquid helium, *Phys. Rev.* **104**, 576-584 (1956).

[22] Shiryayev A.N. *Probability* (Springer-Verlag, Berlin 1984).

[23] Ueltschi D., Geometric and probabilistic aspects of boson lattice models, in *Progr. Probab.* **51**, 363-391 (Birkhäuser, Boston, MA, 2002).

[24] Yang C.N. Concept of Off-Diagonal Long-Range-Order and the quantum phases of liquid *He* and of superconductors, *Rev. Mod. Phys.* **34**, 694-704 (1962).

[25] Zagrebnov V.A. and Bru J.-B., The Bogoliubov model of weakly imperfect Bose gas, *Phys.Rep.* **350**: 291-434 (2001).

[26] Ziff R.M., Uhlenbeck G.E. and Kac M. The ideal Bose-Einstein gas, revisited, *Phys.Rep.* **32**, 169-248 (1977).